

# Weighted Efficient Domination in Classes of $P_6$ -free Graphs

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## Abstract

In a graph  $G$ , an *efficient dominating set* is a subset  $D$  of vertices such that  $D$  is an independent set and each vertex outside  $D$  has exactly one neighbor in  $D$ . The MINIMUM WEIGHT EFFICIENT DOMINATING SET (MIN-WED) problem asks for an efficient dominating set of total minimum weight in a given vertex-weighted graph; the MAXIMUM WEIGHT EFFICIENT DOMINATING SET (MAX-WED) problem is defined similarly. The MIN-WED/MAX-WED is known to be  $NP$ -complete for  $P_7$ -free graphs, and is known to be polynomial time solvable for  $P_5$ -free graphs. However, the computational complexity of the MIN-WED/MAX-WED problem is unknown for  $P_6$ -free graphs. In this paper, we show that the MIN-WED/MAX-WED problem can be solved in polynomial time for two subclasses of  $P_6$ -free graphs, namely for  $(P_6, S_{1,1,3})$ -free graphs, and for  $(P_6, \text{bull})$ -free graphs.

**Keywords:** Graph algorithms; Domination in graphs; Efficient domination; Perfect code;  $P_6$ -free graphs.

## 1 Introduction

Throughout this paper, let  $G = (V, E)$  be a finite, undirected and simple graph with  $n$  vertices and  $m$  edges. For notation and terminology not defined here, we follow [8]. In a graph  $G$ , a subset  $D \subseteq V$  is a *dominating set* if each

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vertex outside  $D$  has some neighbor in  $D$ . An *efficient dominating set* (*e.d.*) is a dominating set  $D$  such that  $D$  is an independent set and each vertex outside  $D$  has exactly one neighbor in  $D$ . Efficient dominating sets were introduced by Biggs [1], and are also called perfect codes, perfect dominating sets and independent perfect dominating sets in the literature. The notion of efficient dominating sets is motivated by various interesting applications such as coding theory and resource allocation in parallel computer networks; see [1, 20]. We refer to [17] for more information on efficient domination in graphs.

The EFFICIENT DOMINATING SET (ED) problem asks for the existence of an efficient dominating set in a given graph  $G$ . The MINIMUM WEIGHT EFFICIENT DOMINATING SET (MIN-WED) problem asks for an efficient dominating set of total minimum weight in a given vertex-weighted graph; the MAXIMUM WEIGHT EFFICIENT DOMINATING SET (MAX-WED) problem is defined similarly.

Clearly, a graph  $G = (V, E)$  has an efficient dominating set if and only if  $(G, w, |V|)$  is a yes instance to the MIN-WED problem, where  $w(v) = 1$ , for every  $v \in V$ , and the MIN-WED problem is equivalent to the MAX-WED problem (see [3]).

The ED problem is known to be *NP*-complete in general, and is known to be *NP*-complete for several restricted classes of graphs such as: bipartite graphs [27], chordal graphs [27], chordal bipartite graphs [24], planar bipartite graphs [24], and planar graphs with maximum degree three [14]. However, ED is solvable in polynomial time for split graphs [11], co-comparability graphs [10, 13], interval graphs [12], circular-arc graphs [12], and for many more classes of graphs (see [3, 9] and the references therein).

Let  $P_k$  denote the chordless path with  $k$  vertices and let  $C_k$  denote the chordless cycle with  $k$  vertices,  $k \geq 3$ . A *hole* is a chordless cycle  $C_k$ , where  $k \geq 5$ . Let  $S_{i,j,k}$  denote a tree with exactly three vertices of degree one, being at distance  $i$ ,  $j$  and  $k$  from the unique vertex of degree three. Note that  $S_{i,j,0}$  is a path on  $i + j + 1$  vertices, while  $S_{1,1,1}$  is called a *claw* and  $S_{1,1,2}$  is called a *chair or fork*. See Figure 1 for some special graphs used in this paper.

In this paper, we focus on the MIN-WED/MAX-WED problem in certain classes of graphs that are defined by forbidden induced subgraphs. If  $\mathcal{F}$  is a family of graphs, a graph  $G$  is said to be  $\mathcal{F}$ -free if it does not contain any induced subgraph isomorphic to any graph in  $\mathcal{F}$ . The ED problem is known to be *NP*-complete for  $(K_{1,3}, K_4 - e)$ -free perfect graphs [23], and for

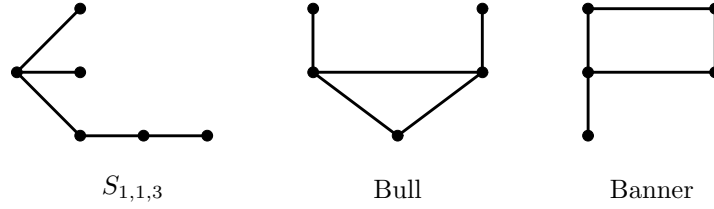


Figure 1: Some special graphs.

$2P_3$ -free chordal graphs [26]. In particular, ED is *NP*-complete for  $P_7$ -free graphs.

Recently, Brandstädt et al. [9] gave a linear time algorithm for solving the MIN-WED/MAX-WED on  $2K_2$ -free graphs, and showed that the MIN-WED/MAX-WED is solvable in polynomial time for  $P_5$ -free graphs. Brandstädt and Le [7] showed that the MIN-WED/MAX-WED is solvable in polynomial time for  $(E, \text{xNet})$ -free graphs, thereby extending the result on  $P_5$ -free graphs. However, the computational complexity of ED is unknown for  $P_6$ -free graphs. Brandstädt et al. showed that ED is solvable in polynomial time for  $(P_6, S_{1,2,2})$ -free graphs [9],  $(P_6, \text{HHD})$ -free graphs, and  $(P_6, \text{house})$ -free graphs [2]. It has also been shown that the MIN-WED/MAX-WED can be solved in polynomial time for  $(P_6, \text{banner})$ -free graphs [18]. We refer to Figure 1 of [3, 9] for the complexity of ED MIN-WED/MAX-WED on several graph classes.

For a graph  $G = (V, E)$  and two vertices  $u, v \in V$ , let  $d_G(u, v)$  denote the *distance* between  $u$  and  $v$  in  $G$ . The *square* of  $G$  is the graph  $G^2 = (V, E^2)$  such that  $uv \in E^2$  if and only if  $d_G(u, v) \in \{1, 2\}$ .

In an undirected graph  $G$ , an *independent set* is a set of mutually non-adjacent vertices. The MAXIMUM WEIGHT INDEPENDENT SET (MWIS) problem asks for an independent set of maximum total weight in the given graph  $G$  with vertex weight function  $w$  on  $V(G)$ . Recently, Brandstädt et al. [3] developed a framework for solving the weighted efficient domination problems based on a reduction to the MWIS problem in the square of the input graph, and is given below.

**Theorem 1 ([3])** *Let  $\mathcal{C}$  be a graph class for which the MWIS problem is solvable in time  $T(|G|)$  on squares of graphs from  $\mathcal{C}$ . Then the MIN-WED/MAX-WED problems are solvable on graphs in  $\mathcal{C}$  in time  $O(\min\{nm + n, n^\mu\} + T(|G^2|))$ , where  $\mu < 2.3727$  is the matrix multiplication exponent [28].*

In this paper, using the above framework, we show that the MIN-WED/MAX-WED problem can be solved in polynomial time in two subclasses of  $P_6$ -free graphs, namely  $(P_6, S_{1,1,3})$ -free graphs and  $(P_6, \text{bull})$ -free graphs. In particular, we prove the following:

- (1) If  $G$  is a  $(P_6, S_{1,1,3})$ -free graph that has an efficient dominating set, then  $G^2$  is  $P_5$ -free (Section 2, Theorem 2).
- (2) If  $G$  is a  $(P_6, \text{bull})$ -free graph that has an efficient dominating set, then  $G^2$  is (hole, banner)-free (Section 3, Theorems 5 and 6).

Since MWIS can be solved in polynomial time for  $P_5$ -free graphs [22] and for (hole, banner)-free graphs (Section 3, Theorem 10), our results follow from (1), (2) and Theorem 1.

Note that the class of  $P_5$ -free graphs is a subclass of  $(P_6, S_{1,1,3})$ -free graphs. Also, note that from the  $NP$ -completeness result for  $K_{1,3}$ -free graphs [23], it follows that for  $S_{1,1,3}$ -free graphs, ED remains  $NP$ -complete. The class of bull-free graphs includes some well studied classes of graphs in the literature such as:  $P_4$ -free graphs, triangle-free graphs, and paw-free graphs.

## 2 Weighted Efficient Domination in $(P_6, S_{1,1,3})$ -free graphs

In this section, we show that the MIN-WED/MAX-WED can be solved efficiently in  $(P_6, S_{1,1,3})$ -free graphs. First, we prove the following:

**Theorem 2** *Let  $G = (V, E)$  be a  $(P_6, S_{1,1,3})$ -free graph. If  $G$  has an efficient dominating set, then  $G^2$  is  $P_5$ -free.*

**Proof.** Let  $G$  be a  $(P_6, S_{1,1,3})$ -free graph having an efficient dominating set  $D$ , and suppose to the contrary that  $G^2$  contains an induced  $P_5$ , say with vertices  $v_1, \dots, v_5$  and edges  $v_i v_{i+1} \in E^2$ ,  $i \in \{1, 2, 3, 4\}$ . Then  $d_G(v_i, v_{i+1}) \leq 2$  for  $i \in \{1, 2, 3, 4\}$  while  $d_G(v_i, v_j) \geq 3$  for  $|i - j| \geq 2$ .

We can assume that  $d_G(v_i, v_{i+1}) = 2$  for all  $i \in \{1, 2, 3, 4\}$  since in all other cases it is easily verified that either  $P_6$  or  $S_{1,1,3}$  is an induced subgraph of  $G$ . For  $i \in \{1, 2, 3, 4\}$ , let  $x_i$  be a common neighbor of  $v_i$  and  $v_{i+1}$ . Note that, by the distance properties, there are no other edges between the vertex sets  $\{x_1, x_2, x_3, x_4\}$  and  $\{v_1, v_2, v_3, v_4, v_5\}$ .

**Claim 2.1**  $x_1 x_4 \in E$ .

*Proof of Claim 2.1:* Suppose to the contrary that  $x_1x_4 \notin E$ . We claim that this implies  $x_1x_3 \notin E$  and  $x_2x_4 \notin E$ : Suppose that  $x_1x_3 \in E$ . Then, if  $x_3x_4 \in E$ ,  $\{v_5, x_4, x_3, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , and if  $x_3x_4 \notin E$ ,  $\{v_5, x_4, v_4, x_3, x_1, v_1\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Thus, under the assumption that  $x_1x_4 \notin E$ , we have  $x_1x_3 \notin E$ , and by symmetry, we have  $x_2x_4 \notin E$ .

Thus, if  $x_1x_4 \notin E$ , the only possible edges among  $\{x_1, x_2, x_3, x_4\}$  are the pairs  $x_ix_{i+1}$ ,  $1 \leq i \leq 3$ , but if all three are edges then  $\{v_1, x_1, x_2, x_3, x_4, v_5\}$  induces a  $P_6$  in  $G$ , and if at least one of the pairs  $x_ix_{i+1}$  is a non-edge, we have an induced  $P_6$  in each case, which is a contradiction.

Hence Claim 2.1 holds.  $\diamond$

**Claim 2.2**  $x_1x_3 \in E$  and  $x_2x_4 \in E$ .

*Proof of Claim 2.2:* Without loss of generality, suppose to the contrary that  $x_2x_4 \notin E$ . Then  $x_1x_2 \in E$  (otherwise,  $\{x_2, v_2, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ ) but then  $\{v_3, x_2, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Thus,  $x_2x_4 \in E$  and by symmetry, also  $x_1x_3 \in E$  holds.

Hence Claim 2.2 is shown.  $\diamond$

By Claims 2.1 and 2.2, we have  $x_1x_3, x_1x_4, x_2x_4 \in E$ . Our next step is:

**Claim 2.3** For all  $i \in \{1, 2, 3, 4\}$ ,  $x_i \notin D$ .

*Proof of Claim 2.3:*

(i) First, suppose to the contrary that  $x_1 \in D$ . Then, since  $D$  is an e.d.,  $v_4, x_4, v_5 \notin D$ . So, there exists  $v'_5 \in D$  such that  $v_5v'_5 \in E$ . Since  $D$  is an e.d.,  $v'_5x_4 \notin E$  and  $v'_5x_1 \notin E$ , and by the distance properties,  $v'_5v_1 \notin E$  and  $v'_5v_2 \notin E$ . Thus,  $\{v'_5, v_5, x_4, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$  which is a contradiction showing that  $x_1 \notin D$ . By symmetry, we obtain  $x_4 \notin D$ .

(ii) Now, suppose to the contrary that  $x_2 \in D$ . Then, since  $D$  is an e.d.,  $v_4, x_4, v_5 \notin D$ . So, there exists  $v'_5 \in D$  such that  $v_5v'_5 \in E$ . Since  $D$  is an e.d.,  $v'_5x_4 \notin E$  and  $v'_5x_2 \notin E$ , and by the distance properties,  $v'_5v_2 \notin E$  and  $v'_5v_3 \notin E$ . Now  $\{v'_5, v_5, x_4, x_2, v_2, v_3\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Hence,  $x_2 \notin D$ . By symmetry, we obtain  $x_3 \notin D$ .

Hence Claim 2.3 holds.  $\diamond$

**Claim 2.4**  $v_2 \notin D$  and  $v_4 \notin D$ .

*Proof of Claim 2.4:* Without loss of generality, suppose to the contrary that  $v_2 \in D$ . If  $v_4 \in D$  then, since  $D$  is an e.d.,  $v_5 \notin D$ . So, there

exists  $v'_5 \in D$  such that  $v_5 v'_5 \in E$ . Again, since  $D$  is an e.d.,  $v'_5$  is not adjacent to  $x_1, x_4, v_2$ , and by the distance properties,  $v'_5 v_1 \notin E$ . Thus,  $\{v'_5, v_5, x_4, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction.

Hence  $v_4 \notin D$  holds. Since by the distance properties,  $v_2 v_4 \notin E$  and by Claim 2.3,  $x_3, x_4 \notin D$ , there exists  $v'_4 \in D$  such that  $v_4 v'_4 \in E$ . Now, if  $v'_4 x_4 \notin E$ , then since  $v'_4 x_1 \notin E$  and  $v'_4 v_2 \notin E$  since  $D$  is an e.d., and since by the distance properties,  $v'_4 v_1 \notin E$ ,  $\{v'_4, v_4, x_4, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Thus,  $v'_4 x_4 \in E$  holds, and by a similar argument,  $v'_4 x_3 \in E$  also holds. This implies  $v_5 \notin D$  since  $D$  is an e.d. Hence there exists  $v'_5 \in D$  such that  $v_5 v'_5 \in E$ . Since  $\{v_5, v'_4, x_3, x_1, v_1, v_2\}$  does not induce an  $S_{1,1,3}$  in  $G$ , we have  $v'_5 \neq v'_4$ . Then, since  $D$  is an e.d.,  $v'_5 x_4 \notin E$ ,  $v'_5 x_1 \notin E$  and  $v'_5 v_2 \notin E$ , and by the distance properties  $v'_5 v_1 \notin E$ . Now,  $\{v'_5, v_5, x_4, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Hence Claim 2.4 is shown.  $\diamond$

Since  $x_1, x_2, v_2 \notin D$  (by Claims 2.3 and 2.4), there exists  $v'_2 \in D$  such that  $v_2 v'_2 \in E$ . Moreover, since  $x_3, x_4, v_4 \notin D$  (by Claims 2.3 and 2.4), there exists  $v'_4 \in D$  such that  $v_4 v'_4 \in E$ . Note that by the distance properties, we have  $v'_2 \neq v'_4$ . Then we prove the following:

**Claim 2.5**  $v_1 \notin D$  and  $v_5 \notin D$ .

*Proof of Claim 2.5:* Without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then, since  $D$  is an e.d.,  $v'_4 x_1 \notin E$ , and by the distance properties,  $v'_4 v_1 \notin E$  and  $v'_4 v_2 \notin E$ . This implies  $v'_4 x_4 \in E$  since  $\{v'_4, v_4, x_4, x_1, v_1, v_2\}$  does not induce an  $S_{1,1,3}$  in  $G$ . Since  $D$  is an e.d.,  $v'_2 x_1 \notin E$  and  $v'_2 x_4 \notin E$ , and by the distance properties,  $v'_2 v_4 \notin E$  and  $v'_2 v_5 \notin E$  but then  $\{v'_2, v_2, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. A symmetric argument shows that  $v_5 \notin D$ .

Hence Claim 2.5 holds.  $\diamond$

**Claim 2.6**  $v'_4 x_1 \in E$  and  $v'_2 x_4 \in E$ .

*Proof of Claim 2.6:* Without loss of generality, suppose to the contrary that  $v'_4 x_1 \notin E$ . By the distance properties,  $v'_4$  is not adjacent to  $v_1$  and  $v_2$ . We first claim that  $v'_4 x_3 \in E$  and  $v'_4 x_4 \in E$ :

If  $v'_4 x_3 \notin E$  then  $\{v'_4, v_4, x_3, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , and if  $v'_4 x_4 \notin E$  then  $\{v'_4, v_4, x_4, x_1, v_1, v_2\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Thus,  $v'_4 x_3 \in E$  and  $v'_4 x_4 \in E$  holds.

Since  $v_1 \notin D$  (by Claim 2.5), there exists  $v'_1 \in D$  such that  $v_1 v'_1 \in E$ . Note that by Claim 2.3,  $v'_1 \neq x_1$ . By the distance properties,  $v'_1 v_4 \notin E$ .

$E$  and  $v'_1v_5 \notin E$ , and since  $D$  is an e.d.,  $v'_1x_4 \notin E$ . If  $v'_1x_1 \notin E$  then  $\{v'_1, v_1, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Thus  $v'_1x_1 \in E$ .

We claim that  $v'_1 \neq v'_2$ : If  $v'_1 = v'_2$  then in the case that  $v'_2x_2 \in E$ ,  $\{v_1, v'_2, x_2, x_4, v_5, v_4\}$  induces an  $S_{1,1,3}$  in  $G$ , and in the other case when  $v'_2x_2 \notin E$ ,  $\{v_1, v'_2, v_2, x_2, x_4, v_5\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Thus,  $v'_1 \neq v'_2$  holds.

Hence, since  $D$  is an e.d.,  $v'_2x_1 \notin E$  and  $v'_2x_4 \notin E$ . Also, by the distance properties,  $v'_2$  is not adjacent to  $v_4$  and  $v_5$ . Now,  $\{v'_2, v_2, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. This finally shows that  $v'_4x_1 \in E$  holds.

By symmetric arguments, we can show  $v'_2x_4 \in E$ .

Hence Claim 2.6 holds.  $\diamond$

Next, we have the following:

**Claim 2.7**  $v'_2v_1 \in E$  and  $v'_4v_5 \in E$ .

*Proof of Claim 2.7:* Without loss of generality, suppose to the contrary that  $v'_2v_1 \notin E$ . Since  $v_1 \notin D$ , there exists  $v'_1 \in D$  such that  $v'_1 \neq v'_2$  and  $v_1v'_1 \in E$ . Then, by Claim 2.6 and since  $D$  is an e.d., we have  $v'_1x_1 \notin E$  and  $v'_1x_4 \notin E$ , and by the distance properties,  $v'_1v_4 \notin E$  and  $v'_1v_5 \notin E$ . Now,  $\{v'_1, v_1, x_1, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction.

By symmetric arguments, we obtain  $v'_4v_5 \in E$ .

Hence Claim 2.7 holds.  $\diamond$

Next we have:

**Claim 2.8**  $v_3 \notin D$ .

*Proof of Claim 2.8:* Suppose to the contrary that  $v_3 \in D$ . Since  $D$  is an e.d., this implies that  $v'_2x_2 \notin E$ ,  $v'_2v_3 \notin E$ , and  $v'_2x_3 \notin E$ . But then  $\{v_1, v'_2, v_2, x_2, v_3, x_3, v_4\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Hence Claim 2.8 holds.  $\diamond$

Thus,  $v_3 \notin D$ . By Claim 2.3,  $x_2, x_3 \notin D$ . Thus, there is  $v'_3 \in D$  with  $v_3v'_3 \in E$ , and by Claim 2.7 and by the distance properties,  $v'_3 \neq v'_2$  and  $v'_3 \neq v'_4$  holds. Then we have the following.

**Claim 2.9**  $v'_3x_2 \in E$  and  $v'_3x_3 \in E$ .

*Proof of Claim 2.9:* Without loss of generality, suppose to the contrary that  $v'_3x_2 \notin E$ . Then, by Claims 2.6 and 2.7,  $\{v'_3, v_3, x_2, x_4, v_4, v_5\}$  induces an  $S_{1,1,3}$  in  $G$ , which is a contradiction. Thus,  $v'_3x_2 \in E$ , and by a symmetric argument, we have  $v'_3x_3 \in E$  which shows Claim 2.9.  $\diamond$

Now, since  $D$  is an e.d., we see that  $\{v_1, v'_2, v_2, x_2, v_3, x_3, v_4\}$  induces a  $P_6$  in  $G$ , which is a contradiction. This finishes the proof of Theorem 2.  $\square$

**Theorem 3** *The MIN-WED/MAX-WED problem can be solved in polynomial time for  $(P_6, S_{1,1,3})$ -free graphs.*

**Proof.** Since the MWIS problem in  $P_5$ -free graphs can be solved in polynomial time [22], Theorem 3 follows by Theorems 1 and 2.  $\square$

### 3 Weighted Efficient Domination in $(P_6, \text{bull})$ -free graphs

Brandstädt et al. [2] showed that if  $G$  is a  $(P_6, \text{bull})$ -free graph that has an efficient dominating set, then  $G^2$  is perfect. Since MWIS can be solved in polynomial time for perfect graphs [16], WED can be solved in polynomial time for  $(P_6, \text{bull})$ -free graphs.

In this section, we show that WED can be solved more efficiently in time  $O(n^2m)$  for  $(P_6, \text{bull})$ -free graphs (which considerably improves the time bound for this graph class).

#### 3.1 Squares of $(P_6, \text{bull})$ -free graphs with e.d. are hole-free

In [2], the following is shown:

**Theorem 4 ([2])** *Let  $G = (V, E)$  be a  $P_6$ -free graph. If  $G$  has an efficient dominating set then  $G^2$  is hole-free.*

Though it directly follows from Theorem 4 that for any  $(P_6, \text{bull})$ -free graph  $G$  with e.d., its square  $G^2$  is hole-free, the structure for  $(P_6, \text{bull})$ -free graphs is more special; we describe this in Theorem 4 and we give a direct proof for Theorem 5 here which is much simpler for the subclass of  $(P_6, \text{bull})$ -free graphs and makes this paper self-contained.

**Theorem 5** *Let  $G = (V, E)$  be a  $(P_6, \text{bull})$ -free graph. Then we have:*

- (i)  $G^2$  is  $C_k$ -free for all  $k \geq 6$ .



(ii) If  $G$  has an efficient dominating set then  $G^2$  is  $C_5$ -free.

**Proof.** Let  $G$  be a  $(P_6, \text{bull})$ -free graph, and let  $H$  denote a hole (isomorphic to  $C_k$ ,  $k \geq 5$ ) in  $G^2$  with vertices  $\{v_1, v_2, \dots, v_k\}$  and edges  $v_i v_{i+1} \in E^2$  (index arithmetic modulo  $k$ ). Then for every  $i \in \{1, 2, \dots, k\}$ , we have  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_i, v_j) \geq 3$  if  $j \notin \{i-1, i+1\}$  and  $j \neq i$ . For  $d_G(v_i, v_{i+1}) = 2$ , let  $x_i$  denote a common neighbor of  $v_i$  and  $v_{i+1}$ .

**Claim 5.1** *If  $\{v_1, v_2, v_3, v_4\}$  induces a  $P_4$  in  $G^2$  with  $d_G(v_i, v_{i+1}) \leq 2$  and  $v_1 v_2 \in E$  then  $v_3 v_4 \in E$ .*

*Proof of Claim 5.1:* If  $v_1 v_2 \in E$  then  $v_2 v_3 \notin E$  since  $d_G(v_1, v_3) \geq 3$ . Thus,  $d_G(v_2, v_3) = 2$ ; let  $x_2$  be a common neighbor of  $v_2$  and  $v_3$ . Now if  $d_G(v_3, v_4) = 2$  and  $x_3$  is a common neighbor of  $v_3$  and  $v_4$  then, since  $\{v_2, v_3, v_4, x_2, x_3\}$  does not induce a bull in  $G$ , we have  $x_2 x_3 \notin E$  but then  $\{v_1, v_2, x_2, v_3, x_3, v_4\}$  induces a  $P_6$  in  $G$ , which is a contradiction. This shows Claim 5.1.  $\diamond$

**Claim 5.2** *For all  $i$ , if  $x_i, x_{i+1}, x_{i+2}$  exist, then  $x_i x_{i+1} \notin E$  and  $x_i x_{i+2} \in E$ .*

*Proof of Claim 5.2:* Without loss of generality, let  $i = 1$ . Since  $\{v_1, x_1, v_2, x_2, v_3\}$  does not induce a bull in  $G$ , we have  $x_1 x_2 \notin E$ , and thus in general,  $x_i x_{i+1} \notin E$ . Now since  $\{v_1, x_1, v_2, x_2, v_3, x_3\}$  does not induce a  $P_6$  in  $G$ , we have  $x_1 x_3 \in E$  and thus in general,  $x_i x_{i+2} \in E$ . This shows Claim 5.2.  $\diamond$

**Claim 5.3** *For all  $i$ , if  $x_i, x_{i+1}, x_{i+3}$  exist, then  $x_i x_{i+3} \in E$ .*

*Proof of Claim 5.3:* Without loss of generality, let  $i = 1$ . By Claim 5.1,  $x_3$  exists. Then since by Claim 5.2 and since  $\{v_1, x_1, x_3, v_4, x_4, v_5\}$  does not induce a  $P_6$  in  $G$ , we have  $x_1 x_4 \in E$ , and thus in general,  $x_i x_{i+3} \in E$ . This shows Claim 5.3.  $\diamond$

*Proof of Theorem 5 (i):* Suppose to the contrary that  $G^2$  contains an even hole  $H$  isomorphic to  $C_{2k}$ ,  $k \geq 3$ . First assume that there is an  $i \in \{1, \dots, 2k\}$  with  $v_i v_{i+1} \in E$ ; without loss of generality, say  $v_1 v_2 \in E$ . Then by the distance properties,  $d_G(v_2, v_3) = 2$ , by Claim 5.1,  $v_3 v_4 \in E$ , and again by the distance properties and by Claim 5.1,  $d_G(v_4, v_5) = 2$  and  $v_5 v_6 \in E$ . Now, since  $\{v_2, x_2, v_3, v_4, x_4, v_5\}$  does not induce a  $P_6$  in  $G$ , we have  $x_2 x_4 \in E$  but then  $\{v_1, v_2, x_2, x_4, v_5, v_6\}$  induces a  $P_6$  in  $G$  which is a contradiction.

Thus, for every  $i \in \{1, \dots, 2k\}$   $d_G(v_i, v_{i+1}) = 2$  holds. Clearly, since  $\{v_i, x_i, v_{i+1}, x_{i+1}, v_{i+2}, x_{i+2}\}$  does not induce a  $P_6$  in  $G$ , we have  $x_i x_{i+2} \in E$  for all  $i \in \{1, \dots, 2k\}$ . For a  $C_6$ , this means that  $\{x_1, x_3, x_5, v_1, v_6\}$  induces a bull in  $G$  which is a contradiction. Now assume that  $k \geq 4$ . Then by Claim 5.2,  $x_1 x_3 \in E$  and  $x_3 x_5 \in E$ , and since  $\{x_1, x_3, x_5, v_1, v_6\}$  does not induce a bull in  $G$ , we have  $x_1 x_5 \notin E$ . By Claim 5.3, we have  $x_1 x_4 \in E$  and  $x_2 x_5 \in E$  but now,  $\{v_1, x_1, x_4, x_2, x_5, v_6\}$  induces a  $P_6$  in  $G$  which is a contradiction. This shows that  $G^2$  is even-hole-free.

Now let  $H$  be an odd hole  $C_{2k+1}$ ,  $k \geq 2$ . First assume that there is an  $i \in \{1, \dots, 2k+1\}$  with  $v_i v_{i+1} \in E$ ; without loss of generality, say  $v_1 v_2 \in E$ . Then by the distance properties,  $d_G(v_2, v_3) = 2$ , by Claim 5.1,  $v_3 v_4 \in E$ , and again by the distance properties and by Claim 5.1,  $d_G(v_4, v_5) = 2$  and  $v_5 v_6 \in E$  and so on, and finally we obtain  $v_{2k+1} v_1 \in E$  which is a contradiction to the distance property  $d_G(v_{2k+1}, v_2) \geq 3$ . Thus, for every  $i \in \{1, \dots, 2k+1\}$   $d_G(v_i, v_{i+1}) = 2$  holds. First assume  $k \geq 3$ . By Claim 5.3, we have  $x_1 x_4 \in E$  and  $x_2 x_5 \in E$  and since  $\{x_1, x_3, x_5, v_1, v_6\}$  does not induce a bull in  $G$ , we have  $x_1 x_5 \notin E$  but now,  $\{v_1, x_1, x_4, x_2, x_5, v_6\}$  induces a  $P_6$  in  $G$  which is a contradiction. This shows that  $G^2$  is  $C_{2k+1}$ -free for  $k \geq 3$ .

*Proof of Theorem 5 (ii):* Finally we consider the case when  $H$  is a  $C_5$  in  $G^2$ ; only in this case we need that  $G$  has an e.d.  $D$ . Recall that for every  $i \in \{1, \dots, 5\}$ , we have  $d_G(v_i, v_{i+1}) = 2$ ,  $x_i x_{i+1} \notin E$  and  $x_i x_{i+2} \in E$ .

**Claim 5.4** *For all  $i \in \{1, 2, \dots, 5\}$ , we have  $v_i \notin D$  and  $x_i \notin D$ .*

*Proof of Claim 5.4:* First, without loss of generality, suppose to the contrary that  $v_1 \in D$ . Then since  $D$  is an e.d.,  $x_1, v_2, x_2, x_4, x_5, v_5 \notin D$ . Again, since  $D$  is an e.d., there exist  $v'_2, v'_5 \in D$  such that  $v'_2 v_2, v'_5 v_5 \in E$ . Note that by the distance properties,  $v'_2 \neq v'_5$ . Then  $\{v'_5, v_5, x_5, v_1, x_1, v_2\}$  induces a  $P_6$  in  $G$ , which is a contradiction. So, for all  $i \in \{1, \dots, 5\}$ ,  $v_i \notin D$ .

Next, without loss of generality suppose that  $x_1 \in D$ . Then, since  $D$  is an e.d.,  $x_4, x_5 \notin D$ . Since  $v_5 \notin D$  and  $D$  is an e.d., there exists  $v'_5 \in D$  such that  $v_5 v'_5 \in E$ . Then by the distance properties, we see that  $\{v'_5, v_5, x_4, x_1, x_3, v_3\}$  induces a  $P_6$  in  $G$ , which is a contradiction. So, for all  $i \in \{1, \dots, 5\}$ , we have  $x_i \notin D$ .  $\diamond$

Since  $D$  is an e.d. for  $G$ , for every  $i \in \{1, \dots, 5\}$ , there exists  $v'_i \in D$  such that  $v_i v'_i \in E$ . Then we have:

**Claim 5.5** *For every  $i, j \in \{1, 2, \dots, 5\}$  with  $i \neq j$ ,  $v'_i \neq v'_j$  holds.*

*Proof of Claim 5.5:* Suppose to the contrary that  $v'_1 = v'_2$ ; clearly  $v'_1 \neq v'_3$ , and by  $v'_1 = v'_2$ ,  $v'_1 \neq v'_5$  holds. Since  $\{v_1, v'_1, v_2, x_2, v_3, v'_3\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_1x_2 \in E$  or  $v'_3x_2 \in E$ . Since  $\{v_2, v'_1, v_1, x_5, v_5, v'_5\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_1x_5 \in E$  or  $v'_5x_5 \in E$ . Since  $\{v'_1, x_2, x_5, v_3, v_5\}$  does not induce a bull in  $G$ , we have  $v'_1x_2 \notin E$  or  $v'_1x_5 \notin E$ ; without loss of generality, assume that  $v'_1x_2 \notin E$  holds. This implies  $v'_3x_2 \in E$ , and since  $\{v_2, x_2, v'_3, v_3, x_3\}$  does not induce a bull in  $G$ , we have  $v'_3x_3 \in E$  but now  $\{v_1, v'_1, v_2, x_2, v'_3, x_3\}$  induces a  $P_6$  in  $G$ , which is a contradiction and thus, Claim 5.5 is shown.  $\diamond$

Now, if  $v'_1x_1 \in E$  and  $v'_1x_5 \in E$  then  $\{v'_2, v_2, x_1, v_1, x_5, v_5\}$  induces a  $P_6$  in  $G$ . Thus, without loss of generality, let us assume that  $v'_1x_1 \notin E$  holds. Since  $\{v'_1, v_1, x_1, v_2, x_2, v_3\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_1x_2 \in E$ . Since  $\{v'_1, x_2, x_5, v_3, v_5\}$  does not induce a bull in  $G$ , we have  $v'_1x_5 \notin E$ . Since  $\{v'_1, x_2, x_4, v_3, v_4\}$  does not induce a bull in  $G$ , we have  $v'_1x_4 \notin E$ .

Now,  $\{v'_1, v_1, x_5, v_5, x_4, v_4\}$  induces a  $P_6$  in  $G$ , which is a contradiction and thus, Theorem 5 is shown.  $\square$

Note that Theorem 5 implies that the square  $G^2$  of any  $(P_6, \text{bull})$ -free graph  $G$  with e.d. is hole-free.

### 3.2 Squares of $(P_6, \text{bull})$ -free graphs with e.d. are banner-free

**Theorem 6** *Let  $G = (V, E)$  be a  $(P_6, \text{bull})$ -free graph. If  $G$  has an efficient dominating set, then  $G^2$  is banner-free.*

*Proof.* Let  $G$  be a  $(P_6, \text{bull})$ -free graph having an efficient dominating set  $D$ , and suppose to the contrary that  $G^2$  contains an induced banner with vertices  $\{v_1, v_2, v_3, v_4, v_5\}$  such that  $\{v_1, v_2, v_3, v_4\}$  form a  $C_4$  in  $G^2$  with edges  $v_i v_{i+1}, v_3 v_5 \in E^2$ , where  $i \in \{1, 2, 3, 4\}$  (index arithmetic modulo 4). Then for  $i \in \{1, 2, 3, 4\}$ ,  $d_G(v_i, v_{i+1}) \leq 2$  and  $d_G(v_3, v_5) \leq 2$ , while  $d_G(v_1, v_3) \geq 3$ ,  $d_G(v_1, v_5) \geq 3$ ,  $d_G(v_2, v_4) \geq 3$ ,  $d_G(v_2, v_5) \geq 3$ , and  $d_G(v_4, v_5) \geq 3$ .

If  $d_G(v_i, v_{i+1}) = 2$  for  $i \in \{1, 2, 3, 4\}$ , then let  $x_i$  denote a common neighbor of  $v_i$  and  $v_{i+1}$ . Moreover, if  $d_G(v_3, v_5) = 2$  then let  $y$  denote a common neighbor of  $v_3$  and  $v_5$ ; we call  $x_i$  and  $y$  *auxiliary vertices*. By the distance properties, we have  $x_i v_j \notin E$  if  $j \notin \{i, i+1\}$  and  $v_i y \notin E$  for  $i \neq 3, i \neq 5$ . Since  $G$  is bull-free,  $x_i x_{i+1} \notin E$  for  $i \in \{1, 2, 3, 4\}$  and  $x_2 y \notin E$ ,  $x_3 y \notin E$  holds.

**Claim 6.1**  $d_G(v_3, v_5) = 2$ .

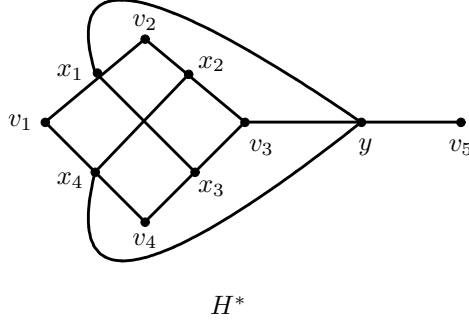


Figure 2: The graph  $H^*$  used in Theorem 6.

*Proof of Claim 6.1:* Suppose to the contrary that  $v_3v_5 \in E$ . Since  $d_G(v_2, v_5) \geq 3$ , we have  $d_G(v_2, v_3) = 2$ , and, since  $d_G(v_4, v_5) \geq 3$ , we have  $d_G(v_3, v_4) = 2$ . So, there exist auxiliary vertices  $x_2$  and  $x_3$ . Since  $d_G(v_2, v_4) \geq 3$ , we have  $d_G(v_1, v_4) = 2$  or  $d_G(v_1, v_2) = 2$ ; without loss of generality, let  $d_G(v_1, v_2) = 2$ . Hence, there exists  $x_1$ . Recall that  $x_1x_2 \notin E$  since  $G$  is bull-free but now  $\{v_5, v_3, x_2, v_2, x_1, v_1\}$  induces a  $P_6$  in  $G$ , which is a contradiction. This shows Claim 6.1.  $\diamond$

Hence,  $d_G(v_3, v_5) = 2$  and the auxiliary vertex  $y$  exists. Since  $d_G(v_2, v_4) \geq 3$ ,  $d_G(v_2, v_3) = 2$  or  $d_G(v_3, v_4) = 2$  holds. We show:

**Claim 6.2**  $d_G(v_2, v_3) = d_G(v_3, v_4) = 2$ .

*Proof of Claim 6.2:* Without loss of generality, suppose to the contrary that  $v_2v_3 \in E$ . Hence  $d_G(v_3, v_4) = 2$  and  $x_3$  exists. Recall that  $x_3y \notin E$  since  $G$  is bull-free. Then, since  $\{v_5, y, v_3, x_3, v_4, v_1\}$  does not induce a  $P_6$  in  $G$ , we have  $d_G(v_4, v_1) = 2$  and thus,  $x_4$  exists.

Moreover, since  $d_G(v_1, v_3) \geq 3$ , we have  $d_G(v_1, v_2) = 2$  and  $x_1$  exists. Recall that  $x_1x_4 \notin E$  since  $G$  is bull-free. Then, by the distance properties,  $\{v_4, x_4, v_1, x_1, v_2, v_3\}$  induces a  $P_6$  in  $G$ , which is a contradiction. By symmetric arguments, we can exclude the case  $v_3v_4 \in E$ .

This shows Claim 6.2.  $\diamond$

Hence, the auxiliary vertices  $x_2$  and  $x_3$  exist. Then since  $G$  is  $P_6$ -free, we easily see that  $d_G(v_1, v_2) = 2$  and  $d_G(v_1, v_4) = 2$ . So, there exist  $x_1$  and  $x_4$ . Recall that  $x_ix_{i+1} \notin E$  for  $i \in \{1, 2, 3, 4\}$  and  $x_2y \notin E$ ,  $x_3y \notin E$  since  $G$  is bull-free.

Then  $x_1x_3 \in E$  and  $x_2x_4 \in E$  since otherwise, either  $\{v_2, x_1, v_1, x_4, v_4, x_3\}$  or  $\{v_4, x_4, v_1, x_2, v_2, x_2\}$  induces a  $P_6$  in  $G$ , which is a contradiction, and

$yx_1, yx_4 \in E$  since otherwise, either  $\{y, v_3, x_3, v_4, x_4, v_1\}$  or  $\{y, v_3, x_2, v_2, x_1, v_1\}$  induces a  $P_6$  in  $G$ , which is a contradiction.

Hence,  $G$  contains  $H^*$  (see Figure 2) as an induced subgraph.

**Claim 6.3**  $x_1, x_2, x_3, x_4, v_2, v_4 \notin D$ .

*Proof of Claim 6.3:*

- (i) Suppose to the contrary that  $x_3 \in D$ . Then since  $D$  is an e.d., we have  $x_1, v_2, x_2 \notin D$ . So, there exists  $v'_2 \in D$  such that  $v_2v'_2 \in E$  and  $v'_2 \neq x_1, x_2$ . Then, since  $\{v'_2, v_2, x_2, v_3, x_3, v_4\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_2x_2 \in E$  but then  $\{v'_2, v_2, x_2, v_3, x_1\}$  induces a bull in  $G$ , which is a contradiction. Hence,  $x_3 \notin D$ , and similarly,  $x_2 \notin D$ .
- (ii) Suppose to the contrary that  $x_4 \in D$ . Then since  $D$  is an e.d., we have  $y, v_5 \notin D$ . So, there exists  $v'_5 \in D$  such that  $v_5v'_5 \in E$  and  $v'_5 \neq y$ . Then since  $D$  is an e.d. and by using the distance properties,  $\{v'_5, v_5, y, x_4, x_2, v_2\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Hence,  $x_4 \notin D$ , and similarly,  $x_1 \notin D$ .
- (iii) Suppose to the contrary that  $v_4 \in D$ . Then since  $D$  is an e.d.,  $v_1 \notin D$ , and by (ii),  $x_1, x_4 \notin D$ . Thus, there exists  $v'_1 \in D$  such that  $v_1v'_1 \in E$  and  $v'_1 \neq x_1, x_4$ . Since  $d_G(v_1, v_3) \geq 3$ , we have  $v'_1v_3 \notin E$ . So, since  $D$  is an e.d.,  $\{v'_1, v_1, x_4, v_4, x_3, v_3\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Hence,  $v_4 \notin D$ , and similarly,  $v_2 \notin D$ .

Thus, Claim 6.3 is proved.  $\diamond$

Since by Claim 6.3,  $x_3, x_4, v_4 \notin D$ , there exists  $v'_4 \in D$  such that  $v_4v'_4 \in E$  and  $v'_4 \neq x_3, x_4$ . Similarly, since by Claim 6.3,  $x_1, x_2, v_2 \notin D$ , there exists  $v'_2 \in D$  such that  $v_2v'_2 \in E$  and  $v'_2 \neq x_1, x_2$ . Moreover, since  $d_G(v_2, v_4) \geq 3$ , we have  $v'_4 \neq v'_2$ .

**Claim 6.4**  $v'_2v_3 \notin E$  and  $v'_4v_3 \notin E$ .

*Proof of Claim 6.4:* Without loss of generality, suppose to the contrary that  $v'_4v_3 \in E$ . Since  $\{v_4, v'_4, v_3, x_2, v_2, v'_2\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_2x_2 \in E$  or  $v'_4x_2 \in E$ .

First assume that  $v'_2x_2 \in E$ . Then, since  $\{x_1, v_2, x_2, v_3, v'_2\}$  does not induce a bull in  $G$ , we have  $v'_2x_1 \in E$ . But now  $\{v_4, v'_4, v_3, x_2, v'_2, x_1\}$  induces a  $P_6$  in  $G$ , which is a contradiction.

Now assume that  $v'_4x_2 \in E$ . Then since  $D$  is an e.d.,  $\{v'_4, v_4, v_3, x_2, v_2\}$  induces a bull in  $G$ , which is a contradiction.

Thus,  $v'_4v_3 \notin E$  and similarly,  $v'_2v_3 \notin E$  which shows Claim 6.4.  $\diamond$

**Claim 6.5**  $v'_2x_2 \notin E$  and  $v'_4x_3 \notin E$ .

*Proof of Claim 6.5:* Without loss of generality, suppose to the contrary that  $v'_4x_3 \in E$ . Since  $\{v'_4, v_4, x_3, v_3, x_4\}$  does not induce a bull in  $G$  and by Claim 6.4, we have  $v'_4x_4 \in E$ . Since  $D$  is an e.d.,  $x_3, v_3, y \notin D$ , and by Claim 6.3,  $x_2 \notin D$ . So, there exists  $v'_3 \in D$  such that  $v_3v'_3 \in E$ . By Claim 6.4,  $v'_3 \neq v'_2$  and  $v'_3 \neq v'_4$ . Since  $d_G(v_1, v_3) \geq 3$ ,  $v'_3v_1 \notin E$ . Now, since  $D$  is an e.d.,  $\{v'_3, v_3, x_3, v_3, v_4, x_4, v_1\}$  induces a  $P_6$  in  $G$ , which is a contradiction. Thus,  $v'_4x_3 \notin E$  and similarly  $v'_2x_2 \notin E$  which shows Claim 6.5.  $\diamond$

Now, since  $\{v'_4, v_4, x_3, v_3, y, v_5\}$  does not induce a  $P_6$  in  $G$  and by Claims 6.4 and 6.5, we have  $v'_4y \in E$ , and similarly, since  $\{v'_2, v_2, x_2, v_3, y, v_5\}$  does not induce a  $P_6$  in  $G$ , we have  $v'_2y \in E$  which contradicts the fact that  $D$  is an e.d. This finally shows Theorem 6.  $\square$

### 3.3 MWIS problem in (hole, banner)-free graphs

In this section, we show that the MWIS problem can be solved in time  $O(n^2m)$  for (hole, banner)-free graphs. To do this, we need the following:

For a vertex  $v \in V(G)$ , the *neighborhood*  $N(v)$  of  $v$  is the set  $\{u \in V(G) \mid uv \in E(G)\}$ , and its *closed neighborhood*  $N[v]$  is the set  $N(v) \cup \{v\}$ . The neighborhood  $N(X)$  of a subset  $X \subseteq V(G)$  is the set  $\{u \in V(G) \setminus X \mid u \text{ is adjacent to a vertex of } X\}$ , and its closed neighborhood  $N[X]$  is the set  $N(X) \cup X$ . Given a subgraph  $H$  of  $G$  and  $v \in V(G) \setminus V(H)$ , let  $N_H(v)$  denote the set  $N(v) \cap V(H)$ , and for  $X \subseteq V(G) \setminus V(H)$ , let  $N_H(X)$  denote the set  $N(X) \cap V(H)$ .

A vertex  $z \in V(G)$  *distinguishes* two other vertices  $x, y \in V(G)$  if  $z$  is adjacent to one of them and nonadjacent to the other. A set  $M \subseteq V(G)$  is a *module* in  $G$  if no vertex from  $V(G) \setminus M$  distinguishes two vertices from  $M$ . The *trivial modules* in  $G$  are  $V(G)$ ,  $\emptyset$ , and all one-vertex sets. A graph  $G$  is *prime* if it contains only trivial modules.

A *clique* in  $G$  is a subset of pairwise adjacent vertices in  $G$ . A *clique separator* (or *clique cutset*) in a connected graph  $G$  is a subset  $Q$  of vertices in  $G$  which induces a complete graph, such that the graph induced by  $V(G) \setminus Q$  is disconnected. A graph is an *atom* if it does not contain a clique separator.

Let  $\mathcal{C}$  be a class of graphs. A graph  $G$  is *nearly*  $\mathcal{C}$  if for every vertex  $v$  in  $V(G)$  the graph induced by  $V(G) \setminus N[v]$  is in  $\mathcal{C}$ .

We first note that prime banner-free graphs are  $K_{2,3}$ -free [6]. We also use the following theorems:

**Theorem 7 ([21])** *Let  $\mathcal{G}$  be a hereditary class of graphs. If the MWIS problem can be solved in time  $O(n^p)$  for prime graphs in  $\mathcal{G}$ , where  $p \geq 1$  is a constant, then the MWIS problem can be solved for graphs in  $\mathcal{G}$  in time  $O(n^p + m)$ .*

**Theorem 8 ([19])** *Let  $\mathcal{C}$  be a class of graphs such that MWIS can be solved in time  $O(f(n))$  for every graph in  $\mathcal{C}$  with  $n$  vertices. Then in any hereditary class of graphs whose atoms are all nearly  $\mathcal{C}$  the MWIS problem can be solved in time  $O(n^2 \cdot f(n))$ .*

In [6], it was shown that prime atoms of (hole, banner)-free graphs are nearly chordal. Applying Corollary 9 in [5] which used an approach for solving MWIS by combining prime graphs and atoms, it was claimed in [6] that MWIS is solvable efficiently for (hole, banner)-free graphs. However, Corollary 9 in [5] is not proven (and thus has to be avoided); a correct way would be to show that atoms of prime (hole, banner)-free graphs are nearly chordal (see also [4] for an example). This will be done in the proof of Theorem 9. Though the proof given here is very similar to that of [6], we carefully analyze and reprove it so as to apply the known theorems.

**Theorem 9** *Every atom of a prime (hole, banner)-free graph is nearly chordal.*

**Proof.** Let  $G$  be a prime (hole, banner)-free graph and let  $G'$  be an atom of  $G$ . We want to show that  $G'$  is nearly chordal, so let us suppose to the contrary that there is a vertex  $v \in V(G')$  such that  $G' \setminus N[v]$  contains an induced  $C_4$ , say  $H$  with vertex set  $\{v_1, v_2, v_3, v_4\}$  and edge set  $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ . For  $i \in \{1, \dots, 4\}$ , we define the following: Let  $Q$  denote the component of  $G \setminus N[H]$  that contains  $v$ , let  $A_i$  denote the set  $\{x \in V(G) \setminus V(H) : |N_H(x)| = i\}$ ,  $A_i^+$  denotes the set  $\{x \in A_i \mid N(x) \cap Q \neq \emptyset\}$ , and  $A^+ = A_1^+ \cup A_2^+ \cup A_3^+ \cup A_4^+$ .

Note that by the definition of  $Q$  and  $A^+$ , we have  $A^+ = N(Q)$ . Hence  $A^+$  is a separator between  $H$  and  $Q$  in  $G$ . Throughout this proof, we take all the subscripts of  $v_i$  to be modulo 4. Then we have the following:

Since  $G$  is banner-free,  $A_1^+ \cup A_3^+ = \emptyset$ , and so  $A^+ = A_2^+ \cup A_4^+$ , where  $A_2^+ = \cup_{i=1}^4 \{x \in A_2 \mid N(x) \cap V(H) = \{v_i, v_{i+1}\}\}$ . Since  $G$  is  $K_{2,3}$ -free and (hole, banner)-free,  $A_4^+$  is a clique. Moreover, since  $G$  is (hole, banner)-free, we see that

- (1)  $A_2^+$  is a clique, and
- (2) every vertex in  $A_2^+$  is adjacent to every vertex in  $A_4^+$ .

So,  $A^+$  is a clique. Since  $A^+$  is a separator between  $H$  and  $Q$  in  $G$ , we obtain that  $V(G') \cap A^+$  is a clique separator in  $G'$  between  $H$  and  $V(G') \cap Q$  (which contains  $v$ ). This contradicts the assumption that  $G'$  is an atom in  $G$ .  $\square$

Using Theorem 9, we now prove the following:

**Theorem 10** *The MWIS problem can be solved in time  $O(n^2m)$  for (hole, banner)-free graphs.*

**Proof.** Let  $G$  be an (hole, banner)-free graph. First suppose that  $G$  is prime. By Theorem 9, every atom of  $G$  is nearly chordal. Since the MWIS problem can be solved in time  $O(m)$  for chordal graphs [15], MWIS can be solved in time  $O(n^2m)$  for  $G$ , by Theorem 8. Then the time complexity is the same when  $G$  is not prime, by Theorem 7.  $\square$

**Theorem 11** *The MIN-WED/MAX-WED can be solved in time  $O(n^2m)$  for  $(P_6, \text{bull})$ -free graphs.*

*Proof of Theorem 11:* Since by Theorem 10, the MWIS problem for (hole, banner)-free graphs can be solved in time  $O(n^2m)$ , Theorem 11 follows by Theorems 5 and 6 and Theorem 1.  $\square$

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